HOMOLOGY SPHERES

BY

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ABSTRACT

An analysis of the homotopy type of spaces with the same homology as the sphere S^n $(n > 1)$ is given. All such spaces are constructed by means of algebraic "invariants" and a certain homology decomposition tower.

1. Introduction

In a previous paper [4] a homotopy classification of spaces with vanishing integral homology was introduced. In general we propose to investigate the following homotopy theoretical question: Given a C.W. complex B, what can be said about maps $f: X \to B$ which induce an isomorphism: $H_*(f,M): H_*(X,M) \to H_*(B,M)$ where M is a module over $\pi_1 B$ of certain type, e.g. trivial module, R-module, etc.

In the present paper we take B to be the n-sphere S^n $(n > 1)$. The space Σ^n will be called a *homology n-sphere* if

$$\tilde{H}_i(\Sigma^n, Z) \approx \begin{cases} 0 \text{ if } i \neq n \\ Z \text{ if } i = n. \end{cases}$$

The integral homology functor $H_*(\, ,Z)$ will be denoted by H_*. A map $X \xrightarrow{f} B$ with $H_* f$ an isomorphism will be called an H-isomorphism. Note that every homology sphere (H-sphere) Σ^n maps to the sphere S^n by a map $f: \Sigma^n \to S^n$ which is an H-isomorphism. In fact, $Z_\infty \Sigma^n$ (see [1]) has the homotopy type of S^n.

Although we state all the results for the integral coefficient case, it can be seen that many of them generalize to other coefficient groups.

By investigating all homology spheres, i.e. maps $\Sigma^n \to S^n$ with acyclic fibre A, one in fact analyses the homotopy type of Aut A, the space of self equivalences of an arbitrary acyclic space A. This goes a long way toward classifying all fibrations

Received June 13, 1972

$A \to X \to X_0$, i.e. all maps $f: X \to X_0$ such that $H_*(f, Z(\pi_1 X_0))$ is an isomorphism. A typical result in this direction is:

THEOREM. *Let A be an acyclic space such that $\pi = \pi_1 A$ acts trivially on $\pi_j A$ for $j > 1$. Then one has a fibration*

$$\mathrm{Aut}_0\ A \to A \to K(\pi/c, 1)$$

where $\mathrm{Aut}_0\ A$ *is the connected component of* Aut A, *and c is the center of π.*

We analyse the homotopy type of a given H-sphere Σ^n by constructing certain Postnikov-like decomposition of Σ^n into a tower of H-spheres. This tower yields "invariants" by means of which one can construct, up to homotopy, all the homology spheres. Certain "geometric" applications and examples are given: an analysis of the Poincaré 3-sphere, and characterization of the $(r-1)$–homotopy type of $2(r+1)$-manifolds which are H-spheres.

Although the nature of the results and proofs is not combinatorial, we work in the category of pointed simplicial Kan complexes since it is more convenient. A general familiarity with [4] is assumed.

1.1. ORGANIZATION OF THE PAPER. In Section 2 we define certain decomposition and the invariants associated with it. We state the main results in Section 3. In Sections 4 and 5 we prove the existence and uniqueness of the homology decomposition tower and prove certain useful properties of the invariants. The main results are then proved in Sections 6 and 7. Section 8 is a supplementary section which renders a certain construction more accessible. Section 9 closes the paper with some geometric examples and applications.

2. Homotopy invariants for homology n-spheres ($n > 1$)

2.1. Let Σ^n be a homology n-sphere $n > 1$. By a *homology decomposition* of Σ^n (or *H-decomposition*) we mean a tower of fibrations

$$\lim_{\substack{\leftarrow \\ k}} E_k = E_\infty \to \dots E_k \xrightarrow{p_k} E_{k-1} \dots \to E_0$$

where E_0 is a simply connected n-sphere, i.e. $E_0 \sim S^n$, together with a (weak) homotopy equivalence $e: \Sigma^n \to E^n_\infty$, which satisfies the following conditions (compare [4, 1.1]):

i) All the maps p_k are homology isomorphisms, in particular E_k is a homology n-sphere.

ii) The k-stage of Σ^n, namely E_k, is j-simple for all $j > k$, i.e., $\pi_1 E_k$ has trivial action of $\pi_j E_k$ for $j > k$.

iii) The fibre of $E_k \to E_{k-1}$ is $(k-1)$-connected.

2.2. UNIQUENESS THEOREM. *Let Σ^n be an H-sphere, and let $\{E_k^i\}$ $i = 1, 2$ be two homology decompositions of Σ^n. Then there exist, in a natural way, a third H-decomposition $\{E_k\}$ and a natural homotopy equivalence $E_k \to E_k^{(i)}$, $1 \leqq k \leqq \infty$, $i = 1, 2$, which commute with all the maps in the towers.*

2.3. EXISTENCE THEOREM. *Let Σ^n be an H-sphere $n > 1$. Then there exists, in a natural way, an H-decomposition*

$$\Sigma_\infty^n \to \dots \Sigma_k^n \stackrel{p_k}{\to} \Sigma_{k-1}^n \to \dots \Sigma_0^n.$$

2.4. REMARK. The main advantage of the H-decomposition 2.3 over the usual Postnikov-Moore decomposition of the map $\Sigma^n \to S^n$ is that the tower 2.3, unlike the Moore-Postnikov tower, supplies natural "invariants" of H-spheres, by means of which all H-decomposition towers and thus all H-spheres can be constructed.

2.5. THE INVARIANTS. We now proceed to define three sets of invariants associated with the H-decomposition tower (2.3):

(i) For each H-sphere Σ^n, let $A_{(n)}$ be the fibre of

$$\Sigma_{n-2}^n \to \Sigma_0^n \sim S^n$$

unless $n = 2$, in which case $A_{(2)}$ will denote the fibre of $\Sigma_1^2 \to \Sigma_0^2 \sim S^2$. Notice that $A_{(n}$ is an acyclic space. In fact, $A_{(n)}$ is an $(n-2)$-stage, i.e., $\pi_1 A_{(n)}$ acts trivially on $\pi_j A_{(n)}$ for $j > n-2$ (unless $n-2$, in which case $A_{(n)}$ is a simple acyclic space [4]). Thus, for every choice of a generator ε for $\pi_n \Sigma_0^n \approx Z$, the boundary maps of $A_{(n)} \to \Sigma^n \to \Sigma_0^n$ will give an element $\eta_\varepsilon(\Sigma^n) \in \pi_{n-1} A_{(n)}$. We regard ε as an element of the multiplicative group $\{\pm 1\}$ which is the group of orientations of Σ_0^n. Thus our first invariant is $\eta_\varepsilon(\Sigma^n)$ (this is the cross-section obstruction of $\Sigma_{n-2}^n \to S^n$).

(ii) We denote by α_k for $k \geqq 1$ the homotopy groups π_k (fibre of p_k) regarded as $\pi_1 \Sigma_k^n = \pi_1 \Sigma_1^n$ modules, whenever $k > 1$. The modules α_k play in the H-decomposition tower the same role played by the homotopy groups in the usual Postnikov tower. Note the α_k's are functorial in Σ^n, and thus may be denoted $\alpha_k \Sigma^n = \alpha_k \Sigma_k^n$. As we shall see later, the following property completely characterizes α_k:

2.6. PROPOSITION. *For all $k \geqq 2$*

$$H_0(\pi, \alpha_k) \approx H_1(\pi, \alpha_k) \approx 0$$

whereas

$$H_1(\alpha_1) \simeq H_2(\alpha_1) \simeq 0.$$

Here π *denotes* $\pi_1 \Sigma_1^n = \pi_1 \Sigma^n$.

(iii) The last set of "invariants" are the k-"*invariants*" of the tower (Σ_k^n). It follows from 2.1 that there is exactly one obstruction to a cross-section of $\Sigma_k^n \xrightarrow{p_k} \Sigma_{k-1}^n$, this obstruction is an element of $H^{k+1}(\Sigma_{k-1}^n, \alpha_k)$ (cohomology with twisted coefficients). This "homology-k-invariant" will be denoted by $h^{k+1}(p_k)$.

3. Construction of H-decomposition towers

We now state the main theorems in terms of the invariants defined above. The theorems describe how to construct all possible towers inductively. Luckily, for the homology n-sphere we can start the induction from $n-2$, rather than from 1 Thus we first construct all possible $(n-2)$-stages in terms of the invariant η (in case $n = 2$, we construct all 1-stages). Then we can proceed to construct the higher stages using the modules α_k and cohomology with local coefficients.

3.1. THEOREM. *The H-spheres* Σ^n $(n > 2)$ *which are j-simple for* $j > n-2$ *are classified by pairs* (A_{n-2}, η) *as follows:*

i) *For any* $(n-2)$*-stage acyclic space* A_{n-2} *and for any element* $\eta_0 \in \pi_{n-1} A_{n-2}$ *there exist a homology n-sphere* Σ^n *and a homotopy equivalence* $A_{n-2} \xrightarrow{e} F$ *where F is the fibre of* $\Sigma^n \to \Sigma_0^n$, *and* $e_{\#}(\eta_0) = \eta_\varepsilon(\Sigma^n) \in \pi_{n-1} F$.

ii) *Any two* $(n-2)$*-stages* Σ^n *and* $'\Sigma^n$ *are homotopy equivalent iff* $F \simeq {'F}$ *where F, 'F are the corresponding acyclic fibres and there exists an isomorphism* $\pi_{n-1} F \to \pi_{n-1}\, {'F}$ *which carries* $\eta_\varepsilon(\Sigma^n)$ *to* $\eta('\Sigma^n)$.

3.2. THEOREM. *The simple homology 2-spheres* Σ^2 (i.e. π_1 *acts trivially on* $\pi_j \Sigma^2$ *for all* $j > 1$) *are classified by the group* α_1, *and the center of* α_1 *(denoted by* $c\alpha_1$*) as follows:*

i) *For each H-sphere* Σ^2, $\eta_\varepsilon(\Sigma^2)$ *lies in* $c\alpha_1$. *Two simple H-spheres* Σ^2 *and* $'\Sigma^2$ *are homotopy equivalent iff* $\alpha_1 \Sigma^2 \cong \alpha_1\, '\Sigma^2$ *and there exists an isomorphism* $c\alpha_1(\Sigma^2) \xrightarrow{\simeq} c\alpha_1('\Sigma^2)$ *which carries* $\eta(\Sigma^2)$ *to* $\eta('\Sigma^2)$.

ii) *For any group* α *which satisfies* $H_1\alpha \cong H_2\alpha \cong 0$ *and any element* $\eta_0 \in c\alpha$ *there exists, in a natural way, a homology sphere* Σ^2 *with* $\alpha_1 \Sigma^2 = \alpha$ *and* $\eta(\Sigma^2) = \eta_0 \in c\alpha_1 \Sigma^2$.

REMARK. Theorems 3.1 and 3.2 in fact classify fibrations over the n-sphere S^n with certain acyclic spaces as fibres. Thus one may consider them as stating the homotopy groups of the space of self-equivalences of A_k, a k-stage acyclic space. For more details see 6.3.

We now continue to construct the H-decomposition tower over Σ_k^n for $k \geqq n-2$ (or $k \geqq 1$ if $n = 2$). Given an H-sphere Σ_k^n which is a k-stage, we have the following classification theorem:

3.5. UNIQUENESS. *Any two $(k+1)$-stages $E_{k+1}^{(i)} \xrightarrow{p_{k+1}^{(i)}} \Sigma_k^n$ $(i = 1,2)$ are fibre homotopy equivalent* [3] *if and only if there exists an isomorphism of $\pi_1\ \Sigma_k$-groups $\alpha_{k+1}\ E_{k+1}^1 \overset{\simeq}{\to} \alpha_{k+1}\ E_{k+1}^2$ which carries $h^{k+2}(p^1)$ to $h^{k+2}(p^2)$.*

3.6. EXISTENCE. *With notation as above given any $\pi_1\ \Sigma_k^n$-module α with $H_0(\pi,\alpha) \simeq H_1(\pi,\alpha) = 0$ $(\pi = \pi_1\ \Sigma_k^n)$ and any (twisted) cocycle $c^{k+2} \in Z^{k+2}(\Sigma_k^n, \alpha)$, there exists a map $p_{k+1}: E_{k+1}^n \to \Sigma_k^n$ which is a homology isomorphism and such that E_{k+1}^n is a $(k+1)$-stage, $\alpha_{k+1}\ E_{k+1}^n \approx \alpha$ as π-modules and $c^{k+2} \in h^{k+2}(p_{k+1})$.*

3.7. H-SPHERES AND ACYCLIC SPACES. There is an intimate connection between the analysis of H-spheres presented here and the analysis of acyclic spaces given in [4]. In fact one has in Fig. 1 a diagram which is homotopy commutative. Thus the

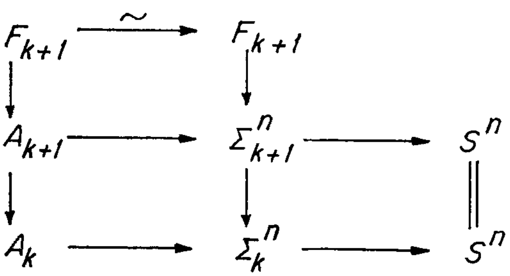

Fig. 1

modules α_k are the same modules α_k as in [4]. In the case of H-spheres one has more invariants which determine the precise nature of the "twist" of $A\Sigma^n$ over S^n.

Let us remark that Theorem 3.6 is somewhat weaker than the corresponding Theorem 1.4(b) in [4], since the fibre map $E_{k+1}^n \to \Sigma_k^n$ is natural only up to homotopy.

4. Proofs of Theorems 2.2 and 2.3

Recall the functor $Z_\infty X$, the nilpotent completion of X as defined by Bousfield and Kan [1]. If $H_1X \approx 0$ then the map $X \to Z_\infty X$ is an H-isomorphism and $Z_\infty X$ is simply connected. In order to define Σ_k^n we start with the Moore-Postnikov decomposition of the map $\Sigma^n \to Z_\infty \Sigma^n$:

$$\Sigma^n \to P_k\Sigma^n \to P_{k-1}\Sigma^n \to \ldots \to Z_\infty\Sigma^n.$$

Define $\Sigma^n_k = P_k\Sigma^n \times_{Z_\infty P_k \Sigma^n} Z_\infty \Sigma^n$, i.e. Σ^n_k is the pull-back in Fig 2. The map g_k is defined as follows:

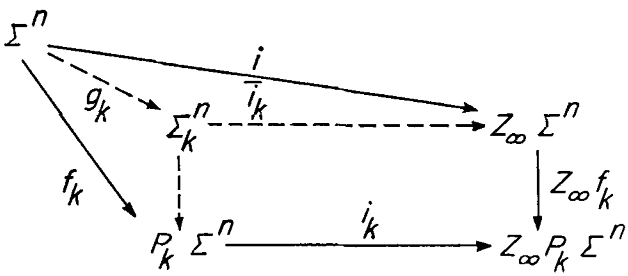

Fig. 2

$$g_k(\sigma) = (f_k(\sigma), i(\sigma)).$$

Note that $Z_\infty f_k$ is a fibre map since f_k is; thus $\Sigma^n_k = \{(a,b) \in P_k\Sigma^n \times Z_\infty\Sigma^n \mid i_k(\sigma) = Z_\infty f_k(b)\}$. Now i_k is an H-isomorphism and $Z_\infty P_k \Sigma^n$ is simply connected; thus $\bar{i}_k$ is also an H-isomorphism, and since i is a homology isomorphism so is g_k.

Since the whole construction is natural one gets a tower

$$\Sigma^n \to \dots \Sigma^n_k \to \Sigma^n_{k-1} \to \dots \Sigma^n_0 = Z_\infty \Sigma^n$$

in which all maps are homology isomorphisms. We must show that properties 2.1 (ii) and (iii) hold. But this is immediate from the corresponding properties of the acyclic decomposition, since $AP_k\Sigma^n$ is the fibre of $\Sigma^n_k \to Z_\infty\Sigma^n$; see Fig. 1 where A is as in [4].

Proof of Uniqueness. Using the functorial tower one gets Fig. 3. Then E_k

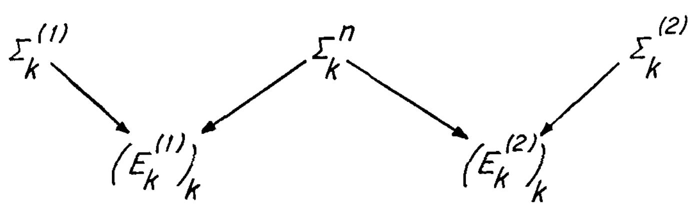

Fig. 3

may be taken as the "total pull-back." Since the maps $E_k \to E_k^{(i)}$ are homology isomorphisms and induce isomorphism on π_j for $j \leq k$ (compare [4, Th. 2.1 (iii), (iv)] and since the spaces are j-simple for $j > k$, these maps are homotopy equivalent.

4.1. Remark. The construction given above is not a step-by-step construction as given in [4]. In Section 8 we give another, inductive construction which shows exactly what homotopy groups and k-invariants come into the construction of Σ_k^n out of $P_k\Sigma^n$.

5. The properties of the invariants

Here we prove two useful propositions:

5.1. LEMMA *Let Σ_1^2 be a simple H-2-sphere. Then $\eta_\varepsilon(\Sigma_1^2) \in c\pi_1\, A\Sigma_1^2$.*

PROOF. Note that $A\Sigma_1^2$ is the fibre of $\Sigma_1^2 \to Z_\infty \Sigma_1^2 \simeq S^2$. But since Σ_1^2 is j-simple for $j > 1$ so is $A\Sigma_1^2$ by [4, Th. 2.1]; thus by the [4, Uniqueness Theorem 4.2] the natural map $A = A\Sigma_1^2 \to AK(\pi_1\Sigma_1^2, 1)$ is a homotopy equivalence. Thus one has in Fig. 4 a diagram of fibrations which, in homotopy, gives the exact ladder depicted in Fig. 5.

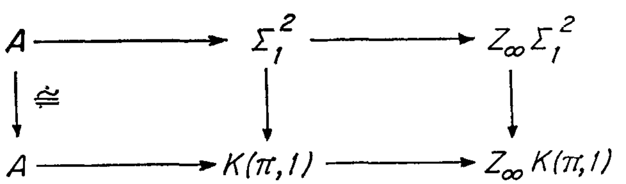

Fig. 4

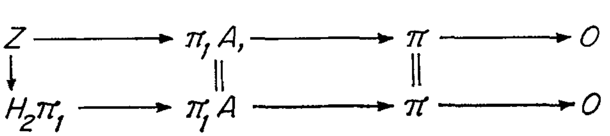

Fig. 5

But π is perfect and $\pi_1 A$ is the universal central extension of π by $H_2\pi$ (compare [5]). Thus η_ε in the center of $\pi_1 A$.

PROOF OF PROPOSITION 2.6. This Proposition follows immediately from the Serre spectral sequence for the fibration

$$F_k \to \Sigma_k^n \overset{p_k}{\to} \Sigma_{k-1}^n$$

for $k \geqq 2$. Since $H_* p_k$ is an isomorphism, and F_k is $(k-1)$-connected, one gets $H_i(\Sigma_{k-1}^n, \pi_k F_k) \simeq 0$ for $i = 0, 1$ which proves the proposition. As for $k = 1$, note that F_1 is an acyclic space; in fact, it is $A(\alpha_1, 1)$. Thus $H_1\alpha_1 = H_2\alpha_2 = 0$.

6. The automorphism of acyclic spaces

For a given $(n-2)$-stage acyclic space A_{n-2}, (or a given $A_1 = A(\alpha_1; 1)$ if $n = 2$) *Theorems 3.1 and 3.2 classify fibrations*

$$A_{n-2} \to \Sigma^n_{n-2} \to S^n$$

$$A(\alpha_1, 1) \to \Sigma_1^2 \to S^2$$

thus in fact computing $\pi_j \operatorname{Aut} A_{n-2}$ for $j \geqq n-1$ and $\pi_j \operatorname{Aut} A(\alpha_1, 1)$ for all $j \geqq 0$. Here Aut X denotes the space of self homotopy equivalences of X. We start with the simpler case 3.2.

6.1. PROOF OF 3.2. We start with 3.2 (ii). Let $H_i\alpha = 0$ $(i = 1, 2)$ and let $c = c\alpha$ be the center of α. Given an element $\bar\eta \in c$, we construct a simple homology 2-sphere as follows: Σ^2 is the pull-back in Fig. 6 in which $\bar\eta$: $S^2 \to K^+$ denotes the homotopy element which corresponds to $\bar\eta$ under the isomorphism

$$\pi_2 K^+ \overset{\simeq}{\to} H_2 K^+ \overset{\simeq}{\to} H_2 K \simeq c.$$

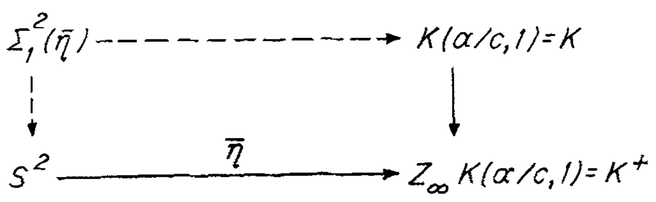

Fig. 6

The first two isomorphisms follow from the general properties of Z_∞ for spaces with perfect fundamental groups. Now, one has an exact sequence $0 \to c \to \alpha \to \alpha/c \to 1$. Thus α/c is perfect. The isomorphism $H_2K = H_2(\alpha/c) \overset{\simeq}{\to} c$ is given by the corresponding exact sequence in homology

$$H_2\alpha \to H_2(\alpha/c) \overset{\simeq}{\to} H_0(\alpha/c, c) \to H_1\alpha$$

since $H_i\alpha = 0$ for $i = 1, 2$ and $H_0(\alpha/c, c) = c$ since the extension is central. Since the universal central extension of α/c is unique, it must be α itself and thus $\alpha_1\ \Sigma_1^2(\bar\eta) \simeq \alpha$. Clearly $\eta_s\ \Sigma_1^2(\bar\eta) = \bar\eta$. This proves (ii). To prove (i), the first claim is Lemma 5.1. As for the rest of (i), note that $\pi_1\ \Sigma_1^2$ is always a quotient of $\alpha_1\ \Sigma_1^2$ by the cyclic subgroup generated by an element of $c\alpha_1 : Z \to \alpha_1 \to \pi_1\ \Sigma^2 \to 0$ is exact. Thus one gets $\alpha_1/c\alpha_1 = \pi_1/c\pi_1$. Now given a homology sphere Σ_1^2 one constructs the map $\Sigma_1^2 \to K(\alpha/c)$ and thus Fig. 7 which is clearly a pull-back diagram. Under the assumption one can construct a self equivalence

$$Z_\infty K(\alpha_1/c, 1) \to Z_\infty K(\alpha_1/c, 1)$$

which carries $\eta(\Sigma^2)$ to $\eta('\Sigma^2)$, thus proving (i).

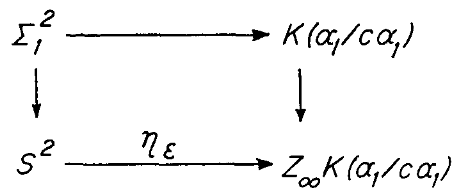

Fig. 7

6.2. Proof. of 3.1. First note that 3.1 states for homology n-spheres $n > 2$," a weaker result than Theorem 3.2 states for $H - 2$-spheres. The proof proceeds along the same line except that for K^+ one uses the complex $Z_\infty P_{n-2} A_{n-2} \simeq Z_\infty P_{n-2}\Sigma^n$ since $P_{n-2} A_{n-2} \simeq P_{n-2}\Sigma^n$. Note that it follows from [4] that $AP_{n-2} A_{n-2} \simeq A_{n-2}$. Thus for each $(n-2)$-stage Σ^n with fibre A_{n-2} over Σ_0^n, one has a pull-back diagram, Fig. 8, derived from the Z_∞-completion of map $\Sigma \to P_{n-2}\Sigma^n$.

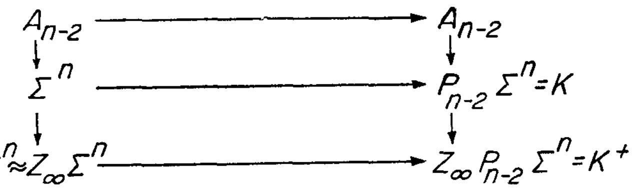

Fig. 8

The main observation is: $\pi_n K^+ \approx H_n K^+ = \pi_{n-1} A_{n-2}$. See 2.2 and 3.2 [4]. We leave it to the reader to check that only Σ^n as above can be derived as a pull-back from K^+, in a unique fashion up to $\mathrm{Aut}(\pi_n K^+)$. Note that for any acyclic space X, $H_{r+1} P_r X \approx 0$ and thus $Z_\infty P_r X$ is $(r+1)$-connected.

6.3. The Automorphisms of A_n. An immediate corollary to Theorems 3.1 and 3.2 is:

Corollary. *Let A_n be an acyclic space in which $\pi_1 A_1$ acts trivially on $\pi_j A_n$, $0 \leqq n < j$. Let* $\mathrm{Aut}\, A_n$ *be the space of self homotopy equivalences of A. Then if $n > 1$, there is map* $\mathrm{Aut}\ A_n \to A_n$ *which induces isomorphism on π_j for all $j > n$. If $n = 1$ (i.e. $A_1 = A(\sigma_1, 1)$), the canonical acyclic space associated with $\sigma = \pi_1 A(\sigma, 1)$ for which $H_i \sigma = 0$, $(i = 1, 2)$:* $\mathrm{Aut}\ A(\sigma, 1)$ *is determined by a* *pull-back diagram in Fig. 9, where $\bar{W}\sigma$ is the Eilenberg-MacLane classifying space $K(\sigma,1)$.*

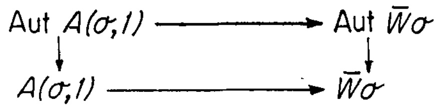

Fig. 9

Note that one has a fibration $K(c\sigma,1) \to \mathrm{Aut}\,\bar{W}\sigma \to \mathrm{Aut}\,\sigma$, where $c\sigma$ is the center of σ. Thus $\pi_1 \mathrm{Aut}\,A(\sigma,1) \approx c\sigma$ and $\pi_j \mathrm{Aut}\,A(\sigma,1) \approx \pi_j A(\sigma,1)$ for all $j > 1$. The above fibre square gives strong ground to the feeling that the acyclic space $A(\sigma,1)$ is the "correct" dual to the aspherical space $K(\sigma,1)$. The homotopy groups of $A(\sigma,1)$ were defined by some authors to be the higher algebraic K-groups of a ring R, when σ is taken to be the Steinberg group $\mathrm{St}(R)$ of the ring R (compare Milnor [3], Quillen and S. Gersten). This is in line with suggestions of Swan and Bass [6] to define the higher K_n by successively annihilating higher homology groups of $\mathrm{St}(R)$. Now let $K^+ = Z_\infty K(GL(R),1)$. It follows from Corollary 6.3 that the universal cover $\tilde{K}^+$ is homotopy equivalent to $\bar{W}\mathrm{Aut}\,A(\mathrm{St}(R),1)$. Thus the higher K-groups K_n for $n \geqq 2$ as defined above, are the groups of homology n-sphere Σ^n with $\alpha_1\Sigma^n \simeq \mathrm{St}(R)$.

7. Construction of the $(k+1)$-stage for $k \geqq n-2$

We now prove the existence Theorem 3.6. The uniqueness will follow easily.

Given a $\pi = \pi_1\Sigma_1^n$-module α, and given a k-stage Σ_k^n with $k \geqq n-2$ (or $k \geqq 1$ if $n = 2$), assume that $H_i(\pi,\alpha) = 0$ for $i = 0, 1$. Let the cocycle C^{k+2} be represented by the map in Fig. 10 where $\phi\colon \pi \to \mathrm{Aut}\,\alpha$ is the given action and where

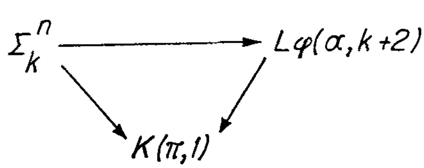

Fig. 10

$L\phi(\alpha, n+1)$ is a classifying space for cohomology with twisted ϕ coefficient, (see [4, 5.1]). Then by pulling back the space of "paths over $K(\pi,1)$" one gets a fibration

$$\begin{array}{ccc} K(\alpha, k+1) & \to & E \\ & & \downarrow{\scriptstyle p'} \\ & & \Sigma_k^n \end{array}$$

such that the natural action is ϕ, and c^{k+2} belongs to the obstruction class in $H^{n+2}(\Sigma_k^n, \alpha)$, to a cross section of p'.

We now want to turn E into a $(k+1)$-stage homology n-sphere. This is done by successively annihilating the higher homology of E. Note that for each fibration like p' with a fibre Eilenberg-MacLane n-space $n > 1$, one has an exact sequence

$$H_1(\pi,\alpha) \to H_{k+2}E \to H_{k+2}B \to H_0(\pi,\alpha) \to H_{k+1}E \to H_{k+1}B \to 0 \quad (B = \Sigma_k^n).$$

Thus since $H_i(\pi,\alpha) = 0$ we get $H_j E \xrightarrow{\approx} H_j B$ for all $j \leqq k+2$, but since $k \geqq n+2$, we see that E is an $H-n$-sphere in dim $\leqq k+2$. Thus $H^{n+1}(E, H_{n+i}E) \approx$ Hom $(H_{n+i}E, H_{n+i}E)$ for $k+2-n \geqq i \geqq 1$ since $\mathrm{Ext}(H_{n+i-1}E, H_{n+i}E) \approx 0$ $(i \geqq 1)$ because $H_nE \approx Z$. Thus there is map $E \to K(H_{k+3}E, k+3)$ which corresponds to the identity map in $H^{k+3}(E, H_{k+3}E) \approx \mathrm{Hom}(H_{k+3}E, H_{k+3}E')$. This map is moreover unique up to homotopy. Define E'_{k+3} to be the fibre of that map; then it is easy to check that E'_{k+3} has the same homology of the n-sphere in dim $\leqq k+3$. Thus one can define a tower E'_j over E, and by taking E^n_{k+1} to be $\lim_{\leftarrow} E'_j$, one gets a homology isomorphism $E^n_{k+1} \xrightarrow{p_{k+1}} \Sigma^n_k$.

Clearly $c^{k+2} \in h^{k+2}(p_{k+1})$. It remains to be proved that $\alpha_{k+1}\, E^n_{k+1} \approx \alpha$ as a π-group. But this follows by comparing $E^n_{k+1} \to \Sigma^n_k$ with the map $A\ \Sigma^n_{k+1} \to A\ \Sigma^n_k$ via Fig. 1, and applying the existence theorem 1.4 in [4].

8. The relative acyclic functor

In constructing Σ_k in the above section, we in fact turned the map $\Sigma^n \to P_k\Sigma^n$ into a homology isomorphism $\Sigma^n \to \Sigma^n_k$, using the completion functor Z_∞. Here we present a simpler construction which does not depend on Z_∞, is more explicit and enables one to gain hold on $\pi_*\Sigma^n_k$.

8.1. THEOREM. *Let X be a connected space with $H_1X \approx 0$. Let $X \to Y$ be a map into another connected space Y. Then there exists, in a natural way, a commutative diagram, Fig.* 11, *with the following properties* (*compare* [4, Th. 2.1]:

i) *The map H_*f is an isomorphism.*

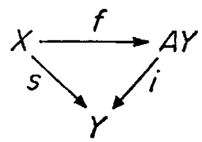

Fig. 11

ii) *The map $AY \to Y$ is universal with respect to maps $X \xrightarrow{g} K \xrightarrow{j} Y$, in which $H_* g$ is an isomorphism (i.e., j factors uniquely through i).*

iii) *The functor A preserves fibre maps, and preserves the j-simplicity of the space (i.e., if Y is j-simple so is AY).*

PROOF. Let $Y_1 \to Y$ be the covering map which corresponds to $P\pi_1 Y$, the maximal perfect subgroup of $\pi_1 Y$. Then one has a unique lifting $X \xrightarrow{f_1} Y_1$ of s. Clearly, $H_1 f_1$ is an isomorphism of the trivial groups. Assume by induction that $f_n\colon X \to Y_n$ has been defined and $H_j f_n$ is an isomorphism for $j \leqq n$. Then we define $f_n'\colon X \to Y_n'$ by the pull-back diagram, Fig. 12. Here Z denotes the reduced free

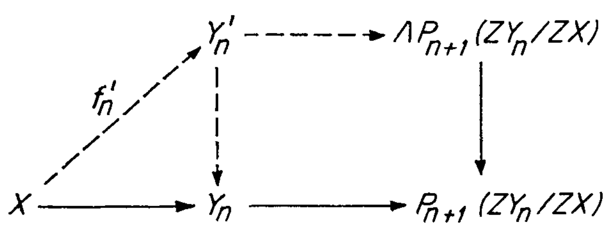

Fig. 12

abelian group functor, P_{n+1} the $(n+1)$-stage of the Postnikov tower and Λ the simplicial path functor. Note that since X maps to the base point in $P_{n+1}(ZY_n/ZX)$, the map f_n lifts to f_n' by: $f_n'(\sigma) = (f_n(\sigma), *)$.

We now claim that $H_j f_n'$ is an isomorphism for $j \leqq n$ and an epimorphism for $j = n+1$. To see this note that P_{n+1} (ZY_n/ZX) has the homotopy type of $K = K(H_{n+1}(Y_n, X), n+1)$. Thus for the fibration $Y_n \to Y_n \to K$ one gets in Fig. 13 an exact ladder, which proves our claim.

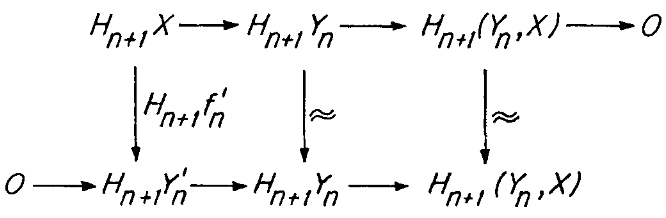

Fig. 13

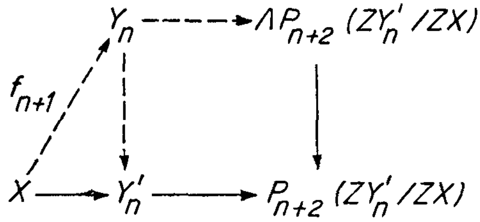

Fig. 14

One proceeds now to define Y_{n+1} by the pull-back diagram, Fig. 14, and prove by similar argument that $H_j f_{n+1}$ is an isomorphism for $j \leqq n+1$. Now the space AY is defined as inverse limit

$$AY = \lim_{\substack{\leftarrow \\ n}} Y_n.$$

Clearly A has all the desired properties (compare [4, Th. 2.1]).

9. Geometrical examples

We start with the well-known Poincaré H-sphere PS^3. This sphere can be derived as a space of orbits of the bi-icosahedral group I^* [8] acting freely on the 3-sphere, or alternatively as $SO(3)/I$ where I is the icosahedral group. One may wonder what is the H-decomposition of Σ^3. This turns out to be a simple question since the universal cover of Σ^3 is S^3.

9.1. Proposition. *Let Σ^n $(n \geqq 2)$ be any H-sphere. If the universal cover of Σ^n is S^n then Σ^n is a simple H-sphere, i.e. $\Sigma^n \sim \Sigma_1^n$.*

Proof. One examines Fig. 15 in which S^n denotes the space $Z_\infty \Sigma^n$ which has the homotopy type of the n-sphere, $\pi = \pi_1 \Sigma^n$. Since $Z_\infty K(\pi, 1)$ is simply connected, and $\pi \to \operatorname{Aut} H_* S^n$ is trivial, $S^n \to F$ must be an H-isomorphism and thus equivalence. Thus $A' \xrightarrow{\sim} AK(\pi, 1)$, which proves the claim. (A denotes the acyclic function [4])

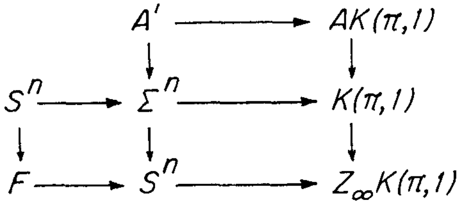

Fig. 15

Now, $H_3 I^* \cong \mathbb{Z}_{120}$, the cyclic group of order 120 (since $|I^*| = 120$). Thus $\pi_2 A(I^*, 1) = \mathbb{Z}_{120}$ and clearly $\eta(PS^3)$ must be a generator. Thus one has:

9.2. Corollary. *Let I^* act freely on S^3. Then $S^3/I^* \in \mathbb{Z}_{32}$ = group of units of $\mathbb{Z}_{120}$.*

9.3 Manifolds. The fundamental group of a closed, compact, smooth n-manifold (which is an) H-sphere for $n \geqq 4$ was characterized by Kervaire to be any finitely related group G with $H_1 G = H_2 G = 0$. It is natural to seek a characterization of the higher homotopy groups of n-manifolds which are H-spheres. Likewise, Kervaire characterized the possible fundamental group of a higher knot. Note that the complement of a knot is a homology 1-sphere (an H-circle). C.T.C. Wall observed [7] that the homotopy type of the complement of a knot: $S^{m-2} \to S^m$ is characterized by a purely homotopy theoretical property, up to the middle dimension:

9.4. Theorem (Wall, Kervaire). *Let (K, L) be a C.W. pair of dimension r and with finite skeletons, such that $K = L \cup_f e^2$ and K is contractible. Then if $m > 2r - 1$, $m \geqq 5$ there is a smooth imbedding of S^{m-2} in S^m, with complement C, and an $(m - r)$-connected map $\psi: C \to L$.*

Notice that L is an almost arbitrary homology 1-sphere of dimension r, the only restrictions are finiteness of skeletons and that $\pi_1 L$ has an element α whose conjugates generate the whole group, (symbolically $w(\pi_1 L) = 1$).

Similarly, one can easily prove:

9.5. Proposition. *Let A be a finite C.W. complex of dimension $r > 2$ such that $\tilde{H}_* A \approx 0$. Then there is a smooth closed manifold M^n for $n > 2r + 1$, which is an $H - n$-sphere and an $(r - 1)$-equivalence $A \to M^n$.*

Thus one can obtain knots and manifolds which are H-spheres by constructing certain finite complexes. In previous papers we showed how to construct all possible acyclic space. However, every (possibly infinite dimensional) locally finite acyclic space gives a finite one as follows:

9.6. Proposition. *Let A be an acyclic C.W. complex with finite skeletons A_d $(d \geqq 0)$. Then for all $d \geqq 2$ there is a finite acyclic complex F and a $(d-1)$-connected map $A_d \to F$.*

Proof. The d-skeleton A_d has vanishing homology through dimension $(d - 1)$; in general, $H_d A_d \neq 0$ but one always has that the map $\pi_d A_d \xrightarrow{h} H_d A_d$ is surjective. To see this, note that coker$(h) = H_d P_{d-1} A_d$, where $P_{d-1} A_d$ is the $(d - 1)$-Postnikov stage of A_d. This is a well-known corollary to the Postnikov tower. Now this general formula for the cokernel of a Hurewicz map implies that $H_d P_{d-1} A \approx 0$. But clearly, $P_{d-1}A \sim P_{d-1}A_d$. We thus can annihilate $H_d A_d$ by adding $(d+1)$-cells to get $F = A_d \cup e^{d+1} \cup \cdots e^{d+1}$ with $\tilde{H}_* F \approx 0$. F is certainly finite, and has the same $(d-1)$-type as A.

Propositions 9.5 and 9.6 together with the classification of acyclic spaces given in [4], combine to generalize the Kervaire theorem about possible fundamental groups of a manifold-H-sphere, and to give complete classification of their homotopy type "up to the middle dimension".

Proposition 9.4 can likewise be used to construct pairs (K, L) as in Proposition 8.4, for which the fibre of $L \to S'$, the given H-isomorphism, is F. One simply takes a Kervaire knot-group, i.e. a finitely presented group G with $H_1 G = \mathbb{Z}$, $H_2 G \approx 0$ and $w(G) = 1$ for which $H_1[G, G] \approx 0$ where $[G, G]$ is the commutator subgroup of G. Then $H_2[G\ G] = 0$. Thus one can take arbitrary A with $\pi_1 A \approx [G, G]$ and construct the mapping torus of a map $A \to A$ with induce on π_1 the natural action of Z on $[G, G]$. Thus, up to the middle dimension one can weaken the Kervaire assumption $\pi_1 K = Z$ to read $[\pi_1, \pi_1]$ is perfect. A more extensive discussion of the homology circle problem will be given in a future paper.

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